

Class notes

of
be
I
to
the

Class meets

MW 11-12:30

My office hr: Tu 3-4

Mr. Berendzen's office hrs, M, Th, _____

In a week or two I will decide what mix of term papers, homework, an exam, & final will be adopted. Problem sets will count, 1-2% of final grade per set.

There are an enormous number of unsolved tractable problems in planetary astronomy for a variety of historical reasons. Some of the I'll touch on in the course of the lectures. By no means are all the major problems solved.

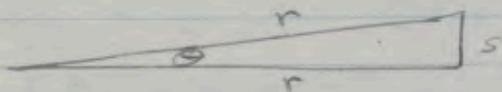
Lecture 1:

We will be concerned primarily with the 2 planets, 3 satellites, + miscellaneous rubble + debris of our own solar system. Except for information about our own planet \oplus , we ~~are~~ ^{have, until recently, been} restricted to remote observations - to the interpretation of electromagnetic radiation reflected or emitted from these objects. Now - for the first time - we have the striking opportunity for direct investigation by space vehicles. Even here, however, the use of flybys + orbiters \Rightarrow remote observation.

We begin today with a short survey of the appearance of the ^{moon and} other planets and some summary of their physical environments - so far as they are known. But to set the stage, consider remote investigations of the planet \oplus .

Consider a satellite at an altitude of 200 - 1000 km. It's orbital speed is ≈ 8 km/sec. Sat's eye it has a telescope w. a modest - 10 cm - aperture. It's diffraction-limited resolution is then

$$\begin{aligned}\theta &= \frac{\lambda}{D} = \frac{5.5 \times 10^{-5} \text{ cm}}{10 \text{ cm}} = 5.5 \times 10^{-6} \text{ rad} \\ &= 2.66 \times 10^5 \text{ sec/rad} \times 5.5 \times 10^{-6} \text{ rad} \\ &= 1.1 \text{ sec arc.}\end{aligned}$$



$$s = r\theta = 10^8 \text{ cm} \times 5.5 \times 10^{-6} \text{ rad} = 5.5 \text{ meters}$$

At 200 km altitude, $s \approx 1$ m.

To obtain this resolution, there must be no blurring by the motion of the satellite, i.e.,

$$t < \frac{10^2 \text{ cm}}{8 \times 10^5 \text{ cm sec}^{-1}} \approx 10^{-4} \text{ sec}$$

which is feasible with fast cameras + film. Thus, it's entirely possible to photograph a man — or at least his late-afternoon

● shadow — from satellite altitude.
Because the turbulent eddies in the atmosphere are near the ground — not near the satellite — you can easily convince yourself by a liver-ore argument that the influence of atmospheric seeing on satellite photography is negligible. The turbulent elements are believed to be some tens of centimeters across.

● Now none of the NASA satellites designed for phys. of ⊕ have this kind of resolution. The 8 TIROS and 1 Nimbus satellites are designed for weather phys. They don't need very high resolution. But there are satellites — in the USAF SAMOS series, according to missile ^{club} ~~trails~~ journals — which do have the ~~required~~ diffraction-limited resolution. I have no SAMOS photos to display here, but I do have some TIROS photos at 1 km resolution.

Composition + structure of solid body. Interior
T, P, ρ , μ , phase. Why are continents where
they are. #

Origin + evolution of the planet.

Existence + character of life.

Water?
①: Atmos. T_s ? Compos. Surface? Origin craters?
Erosion mechanism? ^{Why 50 b's?} Lunar orbiter before Apollo.
Low Albedo.

♀: Synch. rot? Dk. side T?

♀: Clouds. Microwave emission. Surface features.
Source high T_s .

♂: Oblateness. Atmos. T. P. Surface composition. Blue
haze. Life.

♂: Surface? Internal energy sources? GRS?
Bands + belts? Compos. clouds?

η Rings.

♂, ♀; P: ρ .

Satellites ♀.

IV. RADIATIVE ENERGY TRANSPORT

1. Maxwell's Equations and Fresnel's Laws
Planetary
2. Radar Astronomy
3. Particle scattering, diffraction, and rainbows

Reflection of E & M wave from bd.
between 2 media & application to
radar observations of the planets, and polarization
of thermal emission

A. Maxwell's eqns

mk s units

$$\begin{aligned} (1) \quad \nabla \cdot D &= \rho \\ (2) \quad \nabla \cdot B &= 0 \\ (3) \quad \nabla \times E &= -\frac{\partial B}{\partial t} \\ (4) \quad \nabla \times H &= j + \frac{\partial D}{\partial t} \end{aligned} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{field lines ending or not} \\ \text{key terms for wave eqns} \\ \text{oscillating B field} \rightarrow E \\ \rightarrow B \end{array}$$

auxiliary eqns

$$(5) \quad B = \mu H$$

$$(6) \quad D = \epsilon E$$

$$(7) \quad j = \sigma E$$

assume μ, ϵ, σ constant in given medium

B. derivation of wave eqn - let E & B
depend only on one cartesian coord & let $\rho = 0$
then Maxwell become

$$(8) \quad \hat{x} \cdot \frac{\partial D}{\partial x} = 0$$

$$(9) \quad \hat{x} \cdot \frac{\partial B}{\partial x} = 0$$

$$(10) \quad \hat{x} \times \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

$$(11) \quad \hat{x} \times \frac{\partial H}{\partial x} = j + \frac{\partial D}{\partial t}$$

unit vector in
 \hat{x} direction

(8) & (9) show longit components of E & B are indep. of x

& generally these can be neglected as far as wave equation concerned
i.e. they don't propagate in space

use auxiliary relations in (10) & (11)

$$(12) \quad \hat{z} \times \frac{\partial E}{\partial x} = -\mu \frac{\partial H}{\partial t}$$

$$(13) \quad \hat{z} \times \frac{\partial H}{\partial x} = \sigma E + \epsilon \frac{\partial E}{\partial t}$$

take $\left(\frac{\partial}{\partial x} \hat{z} \times\right)$ operating on (12) and use (13) \rightarrow eqn only in E

$$(14) \quad \frac{\partial^2 E}{\partial x^2} - \mu \epsilon \frac{\partial^2 E}{\partial t^2} - \mu \sigma \frac{\partial E}{\partial t} = 0$$

let $E \sim E_0 e^{i\omega t}$

then find ratio 3rd/2nd terms = $\frac{\sigma}{\epsilon \omega}$

if $\frac{\sigma}{\epsilon \omega} \gg 1$ metal, $\frac{\sigma}{\epsilon \omega} \ll 1$ dielectric

a given substance may be metallic at one λ & dielectric at another
eg water metallic at μ wave λ & dielectric at visible λ

Suppose $\frac{\sigma}{\epsilon \omega} \ll 1$, then (14) satisfied by

$$E = g(x - ut) + f(x + ut)$$

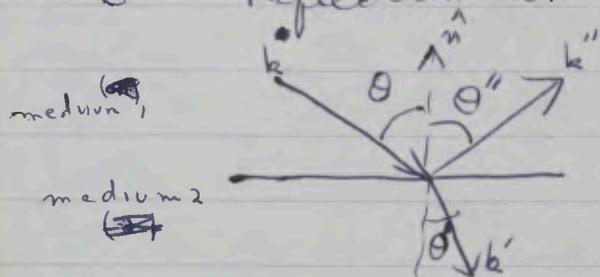
$$u = \frac{1}{\sqrt{\epsilon \mu}} = \frac{c}{\sqrt{\epsilon_r \mu_r}} = \text{phase vel., vel. of prop. wave}$$

~~when include third term of (14)~~ for plane wave constant

$$E = E_0 e^{i(kx - \omega t)}, \quad k = \omega/u$$

when include third term of (14) k becomes complex
with the imaginary part representing a damping of the
wave as it moves further into the media

Reflection at bd. between 2 media of a plane wave



$$(15) \quad E = E_0 e^{i(\hat{k} \cdot \hat{r} - \omega t)}$$

$$(16) \quad H = \frac{\hat{k} \times \vec{E}}{\omega \mu} \quad (\text{from eqn (21)})$$

Bd. conditions (17) $\vec{E}_t(1) = \vec{E}_t(2)$

(18) $H_t(1) = H_t(2)$

\leftarrow or $n \times E(1) = n \times E(2)$
from (3) & (4)
assuming Stoke's theorem
& no surface current

(17) & (18) imply beam has to split into 2 or more parts
 Suppose only refracted. Then ~~since if (17) is satisfied~~ if (17) is satisfied, (18) can't be
 by (16), since $|k_2| \neq |k_1|$

as bd. conditions must be satisfied at all times & all places
~~reflected, refracted, & incident beams must be in phase~~

$$\rightarrow (19) \omega = \omega' = \omega''$$

$$(20) \vec{k} \cdot \vec{r} = \vec{k}' \cdot \vec{r} = \vec{k}'' \cdot \vec{r}$$

from (20) we find $\theta = \theta''$
 $\sin \theta = n \sin \theta' \leftarrow$ Snell's law

Now apply (17) & (18) to amplitudes E_0 & so get reflectivity
 consider 2 cases

A. $E \perp$ to plane of incidence defined by \hat{n} , the normal to
 the surface & \vec{k} , the incident propagation vector

Then (17) becomes

$$(21) \vec{E}_0 + \vec{E}_0'' = \vec{E}_0'$$

$$(18) (22) \vec{E}_0 (\hat{n} \cdot \vec{k}) + \vec{E}_0'' (\hat{n} \cdot \vec{k}) = \vec{E}_0' (n \cdot \vec{k}') \frac{\mu_1}{\mu_2}$$

$$\rightarrow (23) \frac{\vec{E}''_{\perp}}{\vec{E}_{\perp}} = \frac{\mu_2 k_1 \cos \theta - \mu_1 k_2 \cos \theta'}{\mu_2 k_1 \cos \theta + \mu_1 k_2 \cos \theta'}$$

eliminate $\cos \theta'$ by Snell's law ~~or $k_1 \sin \theta = k_2 \sin \theta'$~~

$$\text{reflectivity} = \frac{E'' \times H''}{E \times H} = \frac{E''^2}{E^2}$$

for diel, $k = \frac{\omega}{v} = \frac{\omega}{\frac{c}{\sqrt{\epsilon \mu}}} = \frac{\omega \sqrt{\epsilon \mu}}{c}$ $\text{set } \mu = 1$

$$R_{\perp} = \left[\frac{\cos \theta - (\epsilon - \sin^2 \theta)^{1/2}}{\cos \theta + (\epsilon - \sin^2 \theta)^{1/2}} \right]^2$$

Similarly B. E in plane incidence $\rightarrow H \perp$ plane of incidence

$$R_{\parallel} = \left[\frac{\epsilon \cos \theta - (\epsilon - \sin^2 \theta)^{1/2}}{\epsilon \cos \theta + (\epsilon - \sin^2 \theta)^{1/2}} \right]^2$$

~~If~~ If metal $R \sim 1$

By measuring R , can tell if metal or dielectric
if dielectric can deduce ϵ

Applic:

Radar measurements of R for planets and moon

I Radar equation

consider radar antenna that generates a certain amount of power P_t
which it beams off a planet & then receives the return signal
of power P_r

Em intercepted by planet



$$(1) P_{e_i} = P_t \frac{R_p}{r^2} \leftarrow \text{solid angle planet subtends}$$

$$(2) R_p = \frac{\pi R_p^2}{r^2} \leftarrow \text{distance to planet}$$

~~amount~~ amount received back to antenna

$$(3) P_r = P_s \frac{R_{tel}}{4\pi} d p$$

d is directivity of signal beam off planet
 p is reflectivity
& measure of how preferential back scatter is,

$d = 1$ for isotropic scatter, such as
as case for ~~smooth~~ smooth planet

$$(4) R_{tel} = A_{tel} / r^2$$

so (5) $P_s \sim \frac{1}{r^4} d p$

In applying equation R_{tel} & R_p & in getting d
have to know more than just ^{total} power returned - need
to know how smooth planet is

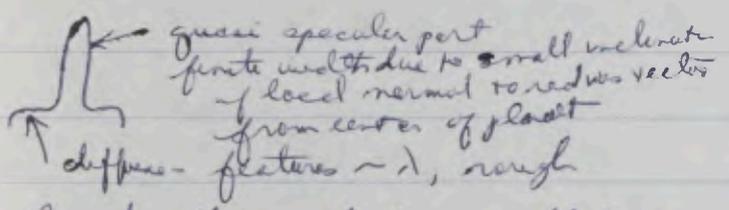
do by range gating 

time return different from different concentric
creeps about subterrastrial point

frequency spectrum = doppler shift 
line of constant freq

together can find how much each area contributes to
returned signal

typical freq. diagram



but to first approximation, almost all signal from small θ ,

so use $R(\theta=0) = \left| \frac{\sqrt{\epsilon} - 1}{\sqrt{\epsilon} + 1} \right|^2$ for dielectric

R ranges 3% for Mars to 10% for Venus

→ surfaces are dielectrics

$\epsilon = \epsilon(\text{composition, porosity})$

for Mars ϵ below ϵ for any nonporous, common mineral
→ material powdered

if know what material it is can get porosity

for Mars deserts thought to be 1/10th

find $g = \rho/\rho_0 = .25$, reasonable value.

Now nonzero surface albedo:

$$x = f e^{c\tau} + g e^{-c\tau}$$

$$y = \frac{u+1}{u-1} f e^{c\tau} + \frac{u-1}{u+1} g e^{-c\tau}$$

\therefore instead of $x_s = 0$, b.c. \rightarrow

$$x_s = A y_s$$

$$\therefore f + g = A \left[\frac{u+1}{u-1} f + \frac{u-1}{u+1} g \right]$$

Assume $g = -Bf$

$$\therefore f - Bf = A \left[\frac{u+1}{u-1} f - \frac{u-1}{u+1} Bf \right]$$

$$f(1-B) = Af \left[\frac{u+1}{u-1} - \frac{u-1}{u+1} B \right]$$

$$A = \frac{\frac{u+1}{u-1} - \frac{u-1}{u+1} B}{1-B}$$

$$1-B + \frac{u-1}{u+1} AB = \frac{u+1}{u-1} A$$

$$1 - \frac{u+1}{u-1} A = B \left(1 - \frac{u-1}{u+1} A \right)$$

$$\therefore B = \frac{1 - \frac{u+1}{u-1} A}{1 - \frac{u-1}{u+1} A} ; A \rightarrow 0 \rightarrow B \rightarrow 1$$

$$\therefore x = f (e^{c\tau} - B e^{-c\tau})$$

$$y = f \left(\frac{u+1}{u-1} e^{c\tau} - \frac{u-1}{u+1} B e^{-c\tau} \right)$$

\therefore effect is to insert a B before $e^{-c\tau}$ in final answers:

$$R = \frac{(u^2 - 1)(e^{\tau_{eff}} - B e^{-\tau_{eff}})}{(u+1)^2 e^{\tau_{eff}} - (u-1)^2 B e^{-\tau_{eff}}}$$

$$T = \frac{(u+1)^2 - B(u-1)^2}{(u+1)^2 e^{\tau_{eff}} - (u-1)^2 B e^{-\tau_{eff}}}$$

Special cases: Conservation scattering, $\Theta_0 \rightarrow 1$, $B \rightarrow 1$.

$$\tau_{eff} = c\tau_1 = \sqrt{3} u (1 - \Theta_0) \tau_1 = \sqrt{3(1 - \Theta_0 + 2\beta\Theta_0)(1 - \Theta_0)} \tau_1 \rightarrow 0$$

$$\therefore e^{\pm \tau_{eff}} \approx 1 \pm \tau_{eff}$$

$$\therefore J = \frac{(u+1)^2 - (u-1)^2}{(u+1)^2(1 + \tau_{eff}) - (u-1)^2(1 - \tau_{eff})}$$

$$= \frac{(u+1)^2 - B(u-1)^2 + u}{[(u+1)^2 - (u-1)^2] + \tau_{eff} [(u+1)^2 + (u-1)^2]} = \frac{4u}{4u + 2\tau_{eff}(u^2 + 1)}$$

$$= \frac{4u}{4u + 2\sqrt{3} u (1 - \Theta_0) \tau_1 (u^2 + 1)} = \frac{1}{1 + \frac{\sqrt{3}}{2} (1 - \Theta_0) \tau_1 (u^2 + 1)}$$

$$u^2 = 1 + \frac{2\beta\Theta_0}{1 - \Theta_0} \Rightarrow u^2 + 1 = 2 \left(\frac{1 - \Theta_0 + \beta\Theta_0}{1 - \Theta_0} \right)$$

$\therefore 1/2\beta$ is roughly the no. of scattering events required to Dopplerize a photon. It's generally several.

$$\therefore J \approx \frac{1}{1 + \sqrt{3} \tau_1 (1 - \Theta_0 + \beta\Theta_0)} \approx \frac{1}{1 + \sqrt{3} \beta \tau_1} \approx \frac{1.16}{2\beta\tau_1 + 1.16}$$

In an exact sol. for $\Theta_0 \rightarrow 1$, $B=1$, and $\exp[-\tau_1/\mu_0] \ll 1$, Pionrowski finds

$$J \approx \frac{1.33}{(1 - \frac{1}{3}\Theta_0) \tau_1 + 1.42}$$

$$R = \frac{(u^2 - 1)[(1 + \tau_{eff}) - (1 - \tau_{eff})]}{(u+1)^2(1 + \tau_{eff}) - (u-1)^2(1 - \tau_{eff})} = \frac{(u^2 - 1)[2\tau_{eff}]}{u^2 + 2u + u^2\tau_{eff} + 2u\tau_{eff} + \tau_{eff} - u^2 + 2u + u^2\tau_{eff} - 2u\tau_{eff} + \tau_{eff}}$$

For the isotropic case, $\beta = 1/2$ and $\Theta_0 = 0$. \therefore we adopt

$$2\beta \approx 1 - \frac{1}{3}\Theta_0$$

$$= \frac{(u^2 - 1)(2\tau_{eff})}{4u + 2u^2\tau_{eff} + 2\tau_{eff}} = \frac{2\beta\Theta_0}{\tau_{eff}(1 - \Theta_0)}$$

Approx. turns out to be good. Should be derivable directly from Legendre expansion.

$$= \frac{2\beta\Theta_0 \tau_{eff}}{2u(1 - \Theta_0) + 2\beta\Theta_0 \tau_{eff}} = \frac{1 - \Theta_0 + \beta\Theta_0}{\tau_{eff} + \frac{(1 - \Theta_0 + \beta\Theta_0)^{1/2} (1 - \Theta_0)^{1/2}}{\beta\Theta_0}}$$

Now nonconform. isotropic scattering $\rightarrow \tau_1 = \infty$ (roughly a deep Rayleigh star.)
 $\Theta_0 \neq 1, \beta = 1/2, \tau_1 = \infty$.

$$\therefore u^2 = \frac{1 - \Theta_0 + 2\beta\Theta_0}{1 - \Theta_0} = \frac{1}{1 - \Theta_0}$$

$\tau_{\text{eff}} \propto \tau_1 = \infty$

$$\therefore R = \frac{(u^2 - 1) e^{-\tau_{\text{eff}}}}{(u+1)^2 e^{\tau_{\text{eff}}}} = \frac{u-1}{u+1} \quad \text{Note for large } \tau_{\text{eff}}, R \text{ unimportant}$$

$$= \frac{1 - \frac{1}{u}}{1 + \frac{1}{u}} = \frac{1 - \sqrt{1 - \Theta_0}}{1 + \sqrt{1 - \Theta_0}}$$

$$T = \frac{4u}{(u+1)^2 e^{\tau_{\text{eff}}}} = 0.$$

Dark at bottom of semi- ∞ narrow coll. stars

Θ_0	R
0.1	0.026
0.5	0.172
0.9	0.520
0.975	0.727

Note very hard to get R above 80%.

Now con., nonisotropic semi- ∞ stars:

$$R = \frac{u-1}{u+1}, \text{ but now } u^2 = \frac{1 - \Theta_0 + 2\beta\Theta_0}{1 - \Theta_0}$$

For fixed Θ_0 , as β declines (more forward scattering), u declines, and R declines. The more forward scattering, the smaller the reflectivity, the greater T . Note that, except for a factor $\sqrt{1 - \Theta_0}$ in τ_{eff} , $\Theta_0 + \beta$ enter into expressions for $R+T$ only as u , i.e., to a significant degree, u can play β off against Θ_0 . Smaller β (more forward scattering) compensates for Θ_0 (less diameter per scattering event); for a given Θ_0 , the latter we trade anisotropy for many scattering events. Note also can to some extent trade by τ_1 for small β : a lot of forward scattering in a thick star is like isotropic scattering in a thick star: the photon direction becomes randomized. Note how surface counts only for the star. Such relations very important for worrying about limb effects: equivalent to increasing τ_1 .

$$\begin{aligned}
 r(\omega_0 \rightarrow 1) &= \frac{\tau_e}{\tau_e + \frac{\sqrt{3}(1-\omega_0)^{1/2}}{\sqrt{3\omega_0}}} \\
 &= \frac{\tau_e}{\tau_e + \frac{\sqrt{3}}{\sqrt{3}}(1-\omega_0)^{1/2}} \\
 &= \frac{\sqrt{3}(2\beta\omega_0)(1-\omega_0)\tau_1}{\sqrt{3}(2\beta\omega_0)(1-\omega_0)\tau_1 + \sqrt{\frac{3}{\beta}}(1-\omega_0)^{1/2}} \\
 &= \frac{\sqrt{6\beta}\tau_1}{\sqrt{6\beta}\tau_1 + \sqrt{2/\beta}} \quad [\text{Check this}] \\
 &= \frac{\sqrt{3\beta}\tau_1}{\sqrt{3\beta}\tau_1 + \sqrt{1/\beta}}
 \end{aligned}$$

$$\text{Cb. } \bar{t} = \frac{1}{1 + \sqrt{3}\beta\tau_1}$$

$$R = \frac{\sqrt{3}\beta\tau_1}{\sqrt{3}\beta\tau_1 + 1}$$

$$\therefore R + \bar{t} = \frac{1 + \sqrt{3}\beta\tau_1}{1 + \sqrt{3}\beta\tau_1} = 1, \text{ as it must for } \omega_0 = 1. \checkmark$$

Now the case of extreme forward scattering, $\beta \approx 0$, $u^2 = 1 + \frac{2\beta\theta_0}{1-\theta_0}$
 Here B is important.

$$R = \frac{\frac{2\beta\theta_0}{1-\theta_0} (e^{\sqrt{3}(1-\theta_0)\tau_1} - B e^{-\sqrt{3}(1-\theta_0)\tau_1})}{A e^{\sqrt{3}(1-\theta_0)\tau_1} - 0}$$

$$= \frac{\beta\theta_0}{2(1-\theta_0)} [1 - B e^{-2\sqrt{3}(1-\theta_0)\tau_1}]$$

For $\beta=0$, $R=0$ indep.
 of B. Why? Because
 no backscattering at
 supra. For $\beta \neq 0$, $R \neq R(B)$.
 Why?

$$T = \frac{2^2 - 0}{2^2 e^{\sqrt{3}(1-\theta_0)\tau_1} - 0} = e^{-\sqrt{3}(1-\theta_0)\tau_1}$$

dep. of $\beta + B$ for β small.

Of Beer's law.

Also exists case of extreme back-scattering, $\beta \approx 1$; $u^2 = 1 + \frac{2\theta_0}{1-\theta_0}$

Optically thin stars: $e^{\pm \tau_{\text{eff}}} \approx 1 \pm \tau_{\text{eff}}$. Similar to $\theta_0 = 1$ case:

$$R = \frac{(u^2-1) [1 + \tau_{\text{eff}} - B + B \tau_{\text{eff}}]}{(u+1)^2 [1 + \tau_{\text{eff}}] - (u-1)^2 B [1 - \tau_{\text{eff}}]}$$

Take case of $\beta = \frac{1}{3}$, $\theta_0 = 0.9$; $u^2 = 1 + \frac{0.9}{0.1} = 10$; $u \approx 3$. $\tau_{\text{eff}} = 0.2$.

$$R = \frac{8 [1.2 - B + 0.2B]}{16 [1.2] - 4B(0.8)} = \frac{8 [1.2 - 0.8B]}{19.2 - 3.2B}$$

$$B = \frac{1 - \frac{4}{3}A}{1 - \frac{2}{7}A} = \frac{1 - 2A}{1 - \frac{1}{2}A}$$

A	B	R
0	1	0.20
0.25	0.57	0.32
1.00	-2	0.47
0.5	0	0.5

Note how sm. τ_{eff} masks
surface albedo variation.
cf. \oplus, \ominus

To summarize for $\beta = \frac{1}{2}$, $\theta_0 = 0.9$,

A	$R(\tau_{eff}=0)$	$R(\tau_{eff}=0.2)$	$R(\tau_{eff}=2)$	$R(\tau_{eff}=20)$
0	0	0.20	0.49	0.50
0.5	0.5	0.50	0.50	0.50
1.0	1.0	0.87	0.51	0.50

Note requirement that for all τ , $R=0.5$ when $A=0.5$. Is there some associated physics?

Limiting cases,

$$(1) \tau_{eff} \ll 1, R = A.$$

$$(2) \tau_{eff} \gg 1, R = \frac{u-1}{u+1}$$

Similarly for transmission,

$$\tau_{eff} \ll 1 \Rightarrow \tau = \frac{(u+1)^2 - B(u-1)^2}{(u+1)^2 + B(u-1)^2} = 1$$

$$\begin{aligned} \tau_{eff} \gg 1 \Rightarrow \tau &= \frac{(u+1)^2 - B(u-1)^2}{(u+1)^2 e^{-\tau_{eff}}} \\ &= \left[1 - \frac{(u-1)^2}{(u+1)^2} B \right] e^{-\tau_{eff}} \\ &= [1 - R^2 B] e^{-\tau_{eff}} \end{aligned}$$

Nominal range for β small (p. 12)

$$R = \frac{\beta \varpi_0}{2(1-\varpi_0)} [1 - B e^{-2\tau_{eff}}]$$

Take $\varpi_0 = 0.9$, $\beta = 0.1$

$$\begin{aligned} \therefore R &= \frac{0.1 \times 0.9}{2 \times 0.1} [1 - B e^{-2\tau_{eff}}] \\ &= 0.45 [1 - B e^{-2\tau_{eff}}] \end{aligned}$$

For $\tau_{eff} \gg 1$ or $B \approx 0$, $R = 0.45$ cf. $R = 0.50$ for $\beta = 0.5$ case

More forward scattering, smaller R , but not by long way.

For $\tau_{eff} \ll 1$, $R = 0.45 [1 - B]$

A	B	R
0	1	0
0.5	0	0.45
1	-2	1.4!

The problem seems to be assuming almost no atmosphere, + strong forward scattering both.

Note effect in gas case of strong absorption: $\varpi_0 = 0.1$, say, $\beta = 1/3$,

$$u^2 = \frac{0.9 + 0.1}{0.9} = 1.1; u = 1.05$$

$$\therefore R = \frac{0.1 (e^{\tau_{eff}} - B e^{-\tau_{eff}})}{4.2 e^{\tau_{eff}} - 0.05 B e^{-\tau_{eff}}}$$

$$\approx \frac{0.1}{4.2} [1 - B e^{-2\tau_{eff}}]$$

$$\therefore R \approx 0.024 [1 - B e^{-2\tau_{eff}}]$$

Quite similar analytically to the strong forward scattering case, except absolute reflectivity cut way down; note $R \ll \varpi_0$.

More generally,

$$R = \frac{(u^2 - 1)(e^{\tau_{off}} - B e^{-\tau_{off}})}{(u+1)^2 e^{\tau_{off}} - (u-1)^2 B e^{-\tau_{off}}}$$

$$u=3, \tau_{off}=2. e^2=7.4; e^{-2}=0.134$$

$$R = \frac{8(7.4 - 0.134 B)}{16 \times 7.4 - 4 \times 0.134 B}$$

$$= \frac{59.2 - 1.07 B}{118.3 - 0.54 B}$$

<u>A</u>	<u>B</u>	<u>R</u>
0	1	0.49
0.5	0	0.50
1.0	-2	0.51

With $\tau_{off} \gg 2$, reduces to $R \approx \frac{u^2 - 1}{(u+1)^2} = \frac{(u+1)(u-1)}{(u+1)^2} = \frac{u-1}{u+1}$

For $u=3$ & this approx, $R \approx \frac{2}{4} = 0.50$. ✓

For very small τ_{off} ,

$$R \approx \frac{(u^2 - 1)[1 - B]}{(u+1)^2 - (u-1)^2 B} = \frac{8(1 - B)}{16 - 4B}$$

$$= \frac{2(1 - B)}{4 - B}$$

<u>A</u>	<u>B</u>	<u>R</u>
0	1	0 ✓
0.5	0	0.5 ✓
1.0	-2	1.0 ✓

Analytically, we must have for this case, $R \approx A$.

Under what circumstances is $u \gg 1$?

$$u^2 = \frac{1 - \alpha_0 + 2\beta\alpha_0}{1 - \alpha_0}$$

$\therefore \beta$ big, i.e., strong backscattering — unphysical. Or $\beta = \frac{1}{2}$, and α_0 very close to 1, i.e., ~~isotropic~~ almost conservative scattering. Then,

$$R = \frac{u^2 (e^{\tau_{eff}} - B e^{-\tau_{eff}})}{u^2 e^{\tau_{eff}} - u^2 B e^{-\tau_{eff}}} = 1$$

$$T = \frac{u^2 - B u^2}{u^2 e^{\tau_{eff}} - u^2 B e^{-\tau_{eff}}} = \frac{1 - B}{e^{\tau_{eff}} - B e^{-\tau_{eff}}}$$

For $\tau_{eff} \gg 1$, $T = e^{-\tau_{eff}} (1-B)$ $R = 1$ is quite astonishing,

since the result is independent of both B and τ_{eff} !

No; for $1 - \alpha_0 \approx 0$ and u big, we require τ , enormous; \therefore we are at conservative, semi- ∞ case, + expect $R = 1$

$R = R(\alpha_0, \beta, \tau, A)$. Any other limiting case? No.

We rewrite our coupled diff. eqs.:

$$\left[\frac{1}{2} \frac{d}{dt} + (1-\alpha_0) + \alpha_0 \beta \right] F_+ = +\alpha_0 \beta F_-$$

$$\left[\frac{1}{2} \frac{d}{dt} - (1-\alpha_0) - \alpha_0 \beta \right] F_- = -\alpha_0 \beta F_+$$

$$\therefore \left[\frac{d}{dt} + 2(1-\alpha_0) + 2\alpha_0 \beta \right] F_+ = 2\alpha_0 \beta F_-$$

$$\left[\frac{d}{dt} - 2(1-\alpha_0) - 2\alpha_0 \beta \right] F_- = -2\alpha_0 \beta F_+$$

$$\therefore \left[\frac{d}{dt} + a \right] F_+ = b F_-$$

$$\left[\frac{d}{dt} + c \right] F_- = e F_+$$

where

$$a = \frac{\sqrt{3}}{2} (1-\alpha_0) + \frac{\sqrt{3}}{2} \alpha_0 \beta$$

$$b = \frac{\sqrt{3}}{2} \alpha_0 \beta$$

$$c = -\frac{\sqrt{3}}{2} (1-\alpha_0) - \frac{\sqrt{3}}{2} \alpha_0 \beta$$

$$e = -\frac{\sqrt{3}}{2} \alpha_0 \beta$$

$$\therefore F_- = \frac{1}{b} \left[\frac{d}{dt} + a \right] F_+$$

$$\therefore \left[\frac{d}{dt} + c \right] \frac{1}{b} \left[\frac{d}{dt} + a \right] F_+ = e F_+$$

$$\left[\frac{d}{dt} + c \right] \left[\frac{d}{dt} + a \right] F_+ = eb F_+$$

$$\therefore \left[\frac{d^2 F_+}{dt^2} + (a+c) \frac{dF_+}{dt} + (ac - eb) F_+ = 0 \right]$$

a homogeneous second-order diff. eq. w. const. coeffs.

For the gen. sol., we try

$$F_+ = A e^{mt}$$

$$\therefore \frac{\delta F_+}{\delta t} = m F_+ ; \quad \frac{\delta^2 F_+}{\delta t^2} = m^2 F_+$$

$$\therefore m^2 F_+ + (a+c) m F_+ + (ac - eb) F_+ = 0$$

$$\therefore m^2 + (a+c)m + (ac - eb) = 0$$

$$\therefore m = \frac{-(a+c) \pm [(a+c)^2 - 4(ac - eb)]^{1/2}}{2}$$

$$-(a+c) = - \left[\frac{\sqrt{3}}{2} (1-\theta_0) + \frac{\sqrt{3}}{2} \theta_0 \beta - \frac{\sqrt{3}}{2} (1-\theta_0) - \frac{\sqrt{3}}{2} \theta_0 \beta \right]$$

$$= - [0] = 0$$

$$\therefore m = \pm \frac{1}{2} [4(eb - ac)]^{1/2}$$

$$= \pm [eb - ac]^{1/2}$$

$$= \pm \left[-\frac{\sqrt{3}}{2} \theta_0 \beta + \frac{\sqrt{3}}{2} \theta_0 \beta - \left\{ \frac{\sqrt{3}}{2} (1-\theta_0) + \frac{\sqrt{3}}{2} \theta_0 \beta \right\} \times \left\{ -\frac{\sqrt{3}}{2} (1-\theta_0) - \frac{\sqrt{3}}{2} \theta_0 \beta \right\} \right]$$

$$= \pm \left[-\frac{3}{4} \theta_0^2 \beta^2 - \frac{3}{4} \left\{ -(1-\theta_0)^2 - \theta_0 \beta (1-\theta_0) - \theta_0 \beta (1-\theta_0) - \theta_0^2 \beta^2 \right\} \right]^{1/2}$$

$$= \pm \left[-\frac{3}{4} \theta_0^2 \beta^2 + \frac{3}{4} (1-\theta_0)^2 + \frac{3}{4} \theta_0 \beta - \frac{3}{4} \theta_0^2 \beta \right. \\ \left. + \frac{3}{4} \theta_0 \beta - \frac{3}{4} \theta_0^2 \beta + \frac{3}{4} \theta_0^2 \beta^2 \right]^{1/2}$$

$$= \pm \left[\frac{3}{4} (1-\theta_0)^2 + \frac{3}{2} \theta_0 \beta - \frac{3}{4} \theta_0^2 \beta \right]^{1/2}$$

$$\therefore F_{\pm} = A e^{m\tau} + B e^{-m\tau}$$

$$\therefore m = \pm \frac{\sqrt{3}}{2} \left[(1 - \alpha_0)^2 + 2\alpha_0\beta(1 - \alpha_0) \right]^{1/2}$$

$$= \pm \frac{\sqrt{3}}{2} (1 - \alpha_0)^{1/2} \left[(1 - \alpha_0) + 2\alpha_0\beta \right]^{1/2}$$

Now this 2nd order diff. eq. demands two b.c.'s. One is that the downward-directed infrared flux at the cloud top is zero — simply an expression of the balance of planetary thermal emission over incoming solar flux in the infrared; i.e.,

$$F_{-}(\tau = \tau_1) = 0$$

We could also have required that $F_{-}(\tau = \tau_1) = S_0$, an approximation a b.c. useful if we consider visible light. Let us in fact carry this condition,

$$F_{-}(\tau = \tau_1) = S_0,$$

and specifying the $S_0 = 0$ later.

$$F_{\downarrow} \quad F_{\uparrow}$$

$$\tau = \tau_1$$

$$\tau = 0$$

It will also be of interest to find $F_{-}(\tau = 0)$ for visible light.

$$F_-(\tau=\tau_1) = S_0 = \frac{1}{b} \left[\frac{d}{d\tau} + a \right]_{\tau=\tau_1} F_+$$

$$\therefore b S_0 = \left. \frac{dF_+}{d\tau} \right|_{\tau=\tau_1} + a F_+(\tau=\tau_1)$$

$$\left. \frac{dF_+}{d\tau} \right|_{\tau=\tau_1} = m A e^{m\tau_1} - m B e^{-m\tau_1}$$

$$\begin{aligned} \therefore b S_0 &= m A e^{m\tau_1} - m B e^{-m\tau_1} + a A e^{m\tau_1} + a B e^{-m\tau_1} \\ &= A e^{m\tau_1} (m+a) + B e^{-m\tau_1} (a-m) \end{aligned}$$

$$\therefore B e^{-m\tau_1} (a-m) = b S_0 - A e^{m\tau_1} (m+a)$$

$$\therefore B = \frac{b S_0}{a-m} e^{m\tau_1} - \frac{A(m+a)}{a-m} e^{2m\tau_1}$$

The second b.c. is

$$F_+(\tau=0) = \text{const.}$$

\therefore , same partial value obtain for the flux upward-directed at the cloud base. For visible light it will be just the value already transmitted through the cloud and reflected off the surface.

$$\therefore F_+(\tau=0) = A + B$$

$$\therefore A = F_+(\tau=0) - B$$

$$= F_+(\tau=0) - \frac{bS_0}{a-m} e^{m\tau} + A \frac{a+m}{a-m} e^{2m\tau}$$

$$\therefore A \left[1 - \frac{a+m}{a-m} e^{2m\tau} \right] = F_+(\tau=0) - \frac{bS_0}{a-m} e^{m\tau}$$

$$\therefore A = \frac{F_+(\tau=0) - \frac{bS_0}{a-m} e^{m\tau}}{1 - \frac{a+m}{a-m} e^{2m\tau}}$$

$$= \frac{(a-m)F_+(\tau=0) - bS_0 e^{m\tau}}{(a-m) - (a+m)e^{2m\tau}}$$

$$A = \frac{(a-m)e^{-m\tau} F_+(\tau=0) - bS_0}{(a-m)e^{-m\tau} - (a+m)e^{+m\tau}}$$

$$B = \frac{bS_0}{a-m} e^{m\tau} - A \frac{a+m}{a-m} e^{2m\tau}$$

$$= \frac{bS_0}{a-m} e^{m\tau} - \frac{(a+m)e^{+m\tau} F_+(\tau=0) + bS_0 \frac{a+m}{a-m} e^{2m\tau}}{(a-m)e^{-m\tau} - (a+m)e^{+m\tau}}$$

$$\therefore F_+ = \left[\frac{(a-m)e^{-m\tau} F_+(\tau=0) - bS_0}{(a-m)e^{-m\tau} - (a+m)e^{+m\tau}} \right] e^{m\tau}$$

$$+ \left[\frac{bS_0}{a-m} e^{m\tau} - \frac{(a+m)e^{+m\tau} F_+(\tau=0) + bS_0 \frac{a+m}{a-m} e^{2m\tau}}{(a-m)e^{-m\tau} - (a+m)e^{+m\tau}} \right]$$

$$\times e^{-m\tau}$$

Now

$$a = \frac{\sqrt{3}}{2} (1 - \theta_0) + \frac{\sqrt{3}}{2} \theta_0 \rho$$

and

$$m = \frac{\sqrt{3}}{2} (1 - \theta_0)^{1/2} \left[(1 - \theta_0)^{1/2} + \frac{\sqrt{3}}{2} \theta_0 \rho \right]^{1/2}$$

$\therefore a+m$ and $a-m$ have no easy simplification.

Still, the eq. on the bottom of p. 9 is usable for any value of $0 \leq \tau \leq \tau_1$, with the usual

Of particular interest is the case $\tau = \tau_1$,

i.e., the energy & flux at the clump top.

$$\begin{aligned} F_+(\tau = \tau_1) &= \frac{(a-m) F_+(\tau = 0) - b S_0 e^{m \tau_1}}{(a-m) e^{-m \tau_1} - (a+m) e^{+m \tau_1}} \\ &+ \frac{b S_0}{a-m} - \frac{(a+m) F_+(\tau = 0) + b S_0 \frac{a+m}{a-m} e^{m \tau_1}}{(a-m) e^{-m \tau_1} - (a+m) e^{+m \tau_1}} \\ &= \frac{b S_0}{a-m} \\ &+ \frac{F_+(\tau = 0) [a-m - (a+m)] + b S_0 e^{m \tau_1} \left[\frac{a+m}{a-m} + 1 \right]}{(a-m) e^{-m \tau_1} - (a+m) e^{+m \tau_1}} \\ &= \frac{b S_0}{a-m} \\ &+ \frac{-2m F_+(\tau = 0) + \frac{2b a S_0}{a-m} e^{m \tau_1}}{(a-m) e^{-m \tau_1} - (a+m) e^{+m \tau_1}} \end{aligned}$$

Let us define

$$u = \left[\frac{(1-\theta_0) + 2\beta\theta_0}{1-\theta_0} \right]^{1/2}$$

$$\therefore (u+1)^2 = \frac{(1-\theta_0) + 2\beta\theta_0}{1-\theta_0} + 2 \left[\frac{(1-\theta_0) + 2\beta\theta_0}{1-\theta_0} \right]^{1/2} + 1$$

$$\therefore a+m = 2(1-\theta_0) + 2\theta_0\beta + 2(1-\theta_0)^{1/2} \left[(1-\theta_0)^{1/2} + 2\theta_0\beta \right]^{1/2}$$

$$\therefore \frac{1}{2}(a+m) = (1-\theta_0) + \theta_0\beta + (1-\theta_0)^{1/2} \left[(1-\theta_0)^{1/2} + 2\theta_0\beta \right]^{1/2}$$

$$\therefore \frac{a+m}{2(1-\theta_0)} = \frac{(1-\theta_0) + \theta_0\beta}{1-\theta_0} + \left[\frac{(1-\theta_0) + 2\beta\theta_0}{1-\theta_0} \right]^{1/2}$$

$$\therefore \frac{a+m}{2(1-\theta_0)} + \frac{\theta_0\beta}{1-\theta_0} + 1 = (u+1)^2$$

$$(u+1)^2 = 2 + \frac{2\beta\theta_0}{1-\theta_0} + 2 \left[\frac{(1-\theta_0) + 2\beta\theta_0}{1-\theta_0} \right]^{1/2}$$

$$(a+m) = \sqrt{2} (1-\theta_0) + \sqrt{2}\theta_0\beta + \sqrt{2} (1-\theta_0)^{1/2} \left[(1-\theta_0)^{1/2} + 2\theta_0\beta \right]^{1/2}$$

$$\frac{a+m}{1-\theta_0} = \frac{\sqrt{2}}{\sqrt{2}} + \frac{\sqrt{2}\theta_0\beta}{1-\theta_0} + \frac{\sqrt{2}}{\sqrt{2}} \left[\frac{(1-\theta_0)^{1/2} + 2\theta_0\beta}{1-\theta_0} \right]^{1/2}$$

$$\therefore \frac{2}{\sqrt{3}}(a+m) = (1-\theta_0)(u+1)^2$$

$$(u+1)^2 = 2 + \frac{2\beta\theta_0}{1-\theta_0} + 2 \left[\frac{(1-\theta_0)^{1/2} + 2\theta_0\beta}{1-\theta_0} \right]^{1/2}$$

$$\frac{a-m}{1-\theta_0} = 2 + \frac{2\beta\theta_0}{1-\theta_0} + 2 \left[\frac{(1-\theta_0)^{1/2} + 2\theta_0\beta}{1-\theta_0} \right]^{1/2}$$

$$\therefore \frac{2}{\sqrt{3}}(a-m) = (1-\theta_0)(u-1)^2$$

$$\text{Also } m = \frac{\sqrt{3}}{\sqrt{2}} u (1-\theta_0)$$

$$\therefore F_+(\tau=\tau_1) = \frac{\sqrt{3} \theta_0 \beta S_0}{(1-\theta_0)(u-1)^2}$$

$$- \frac{\sqrt{3} \sqrt{3} \theta_0 \beta S_0}{(1-\theta_0)(u-1)^2} F_+(\tau=0) - \frac{\sqrt{3} \sqrt{3} \theta_0 \beta S_0}{(1-\theta_0)(u-1)^2} [2(1-\theta_0) + 2\theta_0 \beta] e^{m\tau_1}$$

$$(1-\theta_0)(u-1)^2 e^{-m\tau_1} - (1-\theta_0)(u+1)^2 e^{+m\tau_1}$$

$$= \frac{\sqrt{3} \theta_0 \beta S_0}{(1-\theta_0)(u-1)^2}$$

$$+ \frac{\frac{3}{4} u F_+(\tau=0) - \frac{6}{4} \theta_0 \beta S_0}{(u+1)^2 e^{+m\tau_1} - (u-1)^2 e^{-m\tau_1}} \frac{[(1-\theta_0) + \theta_0 \beta] e^{m\tau_1}}{(u+1)^2 e^{+m\tau_1} - (u-1)^2 e^{-m\tau_1}}$$

$$\therefore F_+(\tau=\tau_1) = v S_0 + \frac{\frac{3}{4} u F_+(\tau=0) - w S_0 e^{m\tau_1}}{(u+1)^2 e^{+m\tau_1} - (u-1)^2 e^{-m\tau_1}}$$

$$\text{where } u = \left[\frac{(1-\theta_0) + 2\theta_0 \beta}{1-\theta_0} \right]^{1/2}$$

$$v = \frac{\sqrt{3} \theta_0 \beta}{(1-\theta_0)(u-1)^2}$$

$$w = \frac{6 \theta_0 \beta [(1-\theta_0) + \theta_0 \beta]}{(1-\theta_0)^2 (u-1)^2}$$

$$\text{and } m = \frac{\sqrt{3}}{4} u (1-\theta_0)$$

Neglecting surface reflection this expression is also valid for visual transparency of the cloud if

we reverse + and - in the vis. $(1-\theta_0) \ll 1$.

Then, for $\log e^{m\tau_1}$

$$u = \left[\frac{2\beta\theta_0}{1-\theta_0} \right]^{1/2}, \text{ a by. no.}$$

$$v = \frac{2\theta_0\beta}{(1-\theta_0)(u^2+2u)} = \frac{2\theta_0\beta}{2\beta\theta_0 + 2(2\beta\theta_0)^{1/2}(1-\theta_0)^{1/2}} = 1$$

$$w = \frac{8\theta_0\beta \left[\frac{\theta_0\beta}{1-\theta_0} \right]}{(1-\theta_0)^2 \frac{2\beta\theta_0}{1-\theta_0}}$$

$$= \frac{4\beta\theta_0}{1-\theta_0}$$

$$\therefore F_+(\tau=\tau_1) = S_0 + \frac{4 \left[\frac{2\beta\theta_0}{1-\theta_0} \right]^{1/2} F_+(u) - \frac{4\beta\theta_0}{1-\theta_0} S_0 e^{-2u(1-\theta_0)\tau_1}}{\frac{2\beta\theta_0}{1-\theta_0} e^{-2u(1-\theta_0)\tau_1}}$$

$$= S_0 + 4 \left(\frac{1-\theta_0}{2\beta\theta_0} \right)^{1/2} F_+(\tau=0) e^{-2u(1-\theta_0)\tau_1} - 2S_0$$

~~At the top of the cloud, the flux is~~ Now S_0 here is the upward directed flux at the cloud bottom, + arises from surface reflection. Roughly $S_0 = S_0' + \tau(1-A_s)$, where τ is the atmospheric transmission, S_0' is the incident solar flux. $F_+(\tau=0) = \text{now also } S_0'$.

$$\therefore \tau = \frac{F_+(\tau=\tau_1)}{S_0'} = -\tau(1-A_s) + 4 \left(\frac{1-\theta_0}{2\beta\theta_0} \right)^{1/2} e^{-\tau_{\text{eff}}}$$

$$\therefore \tau + \tau(1-A_s) = \tau(2-A_s) = 4 \left(\frac{1-\theta_0}{2\beta\theta_0} \right)^{1/2} e^{-\tau_{\text{eff}}}$$

$$\therefore \tau = (2-A_s)^{-1} \left[\frac{8(1-\theta_0)}{2\beta\theta_0} \right]^{1/2} e^{-\tau_{\text{eff}}}$$

for visible light incident from above.

We now return to the informed transparency, where $F_-(\tau=\tau_1) = S_0 = 0$.

The transparency can then be rewritten as

$$\mathcal{L} = \frac{F_+(\tau=\tau_1)}{F_+(\tau=0)} = \frac{4u}{(u+1)^2 e^{\tau_{\text{eff}}} - (u-1)^2 e^{-\tau_{\text{eff}}}}$$

where

$$u = \left[\frac{(1-\theta_0) + 2\beta\theta_0}{1-\theta_0} \right]^{1/2}$$

and

$$\tau_{\text{eff}} = 2u(1-\theta_0)\tau_1$$

For isotropic scattering, $\beta = \frac{1}{2}$, and

$$u_{\text{iso}} = \left[\frac{1-\theta_0 + \theta_0}{1-\theta_0} \right]^{1/2} = (1-\theta_0)^{-1/2},$$

an expression previously derived for random walk.

For isotropic scattering, i.e.,

$$\tau_{\text{eff}} = 2 \frac{(1-\theta_0)}{(1-\theta_0)^{1/2}} \tau_1 = 2(1-\theta_0)^{1/2} \tau_1.$$

In the asymptotic limit of ^{conservative} pure scattering,

$$\theta_0 = 1,$$

$$u \rightarrow \left[\frac{2\beta\theta_0}{1-\theta_0} \right]^{1/2} \approx \left[\frac{2\beta\theta_0}{1-\theta_0} \right]^{1/2}$$

and $\tau_{\text{eff}} \rightarrow 2(2\beta\theta_0)^{1/2} (1-\theta_0)^{1/2} \tau_1 \rightarrow 0$

$$(u+1)^2 \rightarrow u^2 + 2u$$

$$(u-1)^2 \rightarrow u^2 - 2u$$

$$\therefore \phi \rightarrow \frac{4u}{(u^2_{out})^{T_{off}} - (u^2_{in})^{-T_{off}}}$$

$$e^{T_{off}} \rightarrow 1 + T_{off}$$

$$e^{-T_{off}} \rightarrow 1 - T_{off}$$

$$\therefore \phi \rightarrow \frac{4}{(u+2)(1+T_{off}) - \frac{(u-2)}{1-T_{off}}}$$

$$= \frac{4}{u + uT_{off} + 2 + 2T_{off} - \frac{u-2}{1-T_{off}}}$$
~~$$= \frac{4}{u + uT_{off} + 2 + 2T_{off} - \frac{u-2}{1-T_{off}}}$$~~

$$= \frac{4}{2uT_{off} + 4} = \frac{2}{uT_{off} + 2}$$

$$\therefore \phi \rightarrow \frac{1}{2\beta\tau_1 + 1}$$

We can eq. our eq. for ϕ at $\log \tau$ in a pure scattering regime to better understand p.

According to Piontavalis, the flux of ^{trapped} neutrons ~~emerging~~ from the bottom of a thick plane-parallel conventional scattering layer is

$$\phi = \mu_0 \left[e^{-\tau_1/\mu_0} + \frac{H(\mu_0)}{\sqrt{3}(1-\beta_1/3)\tau_1 + 1.42 + 1A/3(1-A)} \right]$$

where the plane function has been expanded in

Legendre polynomials w. coefficients β_n .

Here A is the ground albedo. The Pictrovalis derivation is only valid at $\lambda = \tau_1$, where, for our purposes, the expression reduces to

$$k \approx \frac{\mu_0 H(\mu_0)}{\sqrt{3} (1 - \Theta, 1/3) \tau_1}$$

Now take $\mu_0 = 0.5$, $H(\mu_0) = 2.0$.

$$\therefore k \approx \frac{1}{\sqrt{3} (1 - \Theta, 1/3) \tau_1}$$

$$\therefore \sqrt{3} (1 - \Theta, 1/3) \approx 2\beta$$

$$\beta \approx 0.87 (1 - \Theta, 1/3)$$

$$\text{where } p(\Theta) = 1 + \Theta, \cos \Theta,$$

} for conservative scattering

When $\Theta_1 = 0$, should have $\beta = \frac{1}{2}$. correct factor is

$$\boxed{\beta = 0.5 (1 - \Theta, 1/3), \text{ cons. scatt.}}$$

For isotropic conservative scattering, Sobolev shows that $k = \frac{1}{\tau_1}$.

$$k = \frac{1}{\tau_1}$$

anisotropic

Multifac scattering suggests the multiplication of τ_1 by a factor $(1 - \Theta, 1/3)$. This is the only

place, according to Pictrowsky, that anisotropic scattering enters. Sobolev's complete expression is

$$k = \frac{1.25}{\tau_1 + 1.4}$$

Pictrowsky's results suggest this be replaced by

$$k = \frac{1.25}{\tau_1 (1 - 0.13) + 1.4}$$

In general, the effect of anisotropic scattering will be the introduction of a factor g

$$k = \frac{1.25}{(\tau_1/g) + 1.4}$$

g is roughly the number of scattering events required for the photon to effectively reverse direction. For isotropic scattering, $g = 1$. For anisotropic scattering, $g > 1$, and a larger τ_1 is required to achieve the same k .

But we have also found that for ^{an}isotropic scattering, $\frac{1}{2} [1 + \mathcal{R}]$ more radiation is back-scattered during a single ^{an}isotropic scattering event than during a single isotropic event.

Thus after $\frac{1}{2}[1+\mathcal{K}]$ anisotropic scattering events the photon ~~is~~^{would be} effectively deviated through 90° if multiple anisotropic scattering could be considered a single sum of single anisotropic scattering events. Actually multiple scattering is more efficient in direction reversal than a single sum of single scattering events, because the scattering are not independent — i.e., a deviation in direction in an early scattering event enhances the probability of ~~of~~^{is not} 90° deviation in the next scattering event. As an expression for the number of such events in multiple anisotropic scattering, the expression

$$g = \frac{a\mathcal{K} + 1}{a + 1}$$

suggests itself, where $a \leq 1$. What is a ?

For the phase function $p(\Theta) = 1 + \cos(\Theta)$, $\mathcal{K} = 1$,

and $\mathcal{K} = 3$. \therefore from

$$g = \frac{1}{1 - 2/3} = \frac{a\mathcal{K} + 1}{a + 1}$$

we find

$$\frac{1}{1 - \frac{1}{3}} = \frac{3a+1}{a+1} = \frac{3}{2}$$

$$\therefore 3a+3 = 6a+2$$

$$3a = 1 \text{ and } a = 0.33.$$

For 6μ water drops in visible light, Stober finds $\frac{1}{3} \theta_1 = \frac{7}{8}$, and Demmelij finds $R \approx 25$.

$$\therefore \frac{1}{1 - \frac{7}{8}} = \frac{25a+1}{a+1} = 8$$

$$8(a+1) = 25a+1 = 8a+8$$

$$17a = 7, a = 0.41.$$

For these widely differing cases, a τ must ^{probably} be roughly constant. We choose $a \approx 0.37$ in subsequent discussion, although we will not be applying these eqs. to real phase functions very different from those we have just described in deriving the value of a . Since $\frac{1}{3} \approx 0.37$, we may write

$$g = \frac{1}{1 - \theta_{1,13}} \approx \frac{R + e}{1 + e} = \frac{R + 2.7}{3.7}$$

Thus for $p(\Theta) = 1 + \cos \Theta$, we require about $\frac{3}{2}$ scattering events for 90° deflection, but for μ water droplets in visible light, about $\frac{27.7}{3.7} = 7.5$ scattering events.

If we include absorption the ^{average} photon penetration ~~is~~ average path length into the cloud. Thus for reasonably thick clouds, the fraction of photon which emerges at $t = t_0$ must be very small.

Now let us compare the solution we have obtained for the transmission through multiple and anisotropically but consecutively scattering clouds in the Salton - Schuyler Hill approximation

$$t = \frac{1}{2\beta\tau_0 + 1}$$

with the results of the foregoing discussion based on Salton and Pictorialis,

$$t = \frac{1.25}{(\tau_0, 1.9) + 1.4} \quad \text{van de Hulst says } 1.33$$

$$= \frac{1}{(\tau_0, 1.25) + 1.1}$$

The difference between 1.0 and 1.1 is negligible,

and we find

$$2\beta\tau_1 = \frac{\tau_1}{1.25g}$$

$$\therefore \beta = \frac{1}{2.5g} = 0.4/g = \frac{0.4 \times 3.7}{25 + 2.7} = \frac{1.5}{25 + 2.7}$$

$$\boxed{\beta = \frac{0.4}{g} = \frac{1.5}{25 + 2.7}}$$

This 0.4 should be compared with the 0.5 required for $\beta = \frac{1}{2}$ for isotropic scattering ($\omega_0 = 0$).

If we increase by 20%, we have

$$\boxed{\beta = \frac{0.5}{g} = \frac{1.8}{25 + 2.7}}$$

which should be good to about 10%.

We can now seek other asymptotic solutions of the eq. on p. 10. Consider a very effective optical depth,

$$\tau_{\text{eff}} \gg 1$$

and almost conservative scattering

$$1 - \omega_0 \ll 1.$$

$$\text{Then, } u \approx \left[\frac{2\beta\omega_0}{1-\omega_0} \right]^{1/2} = \left[\frac{2 \frac{1}{2.5}}{1-\omega_0} \right]^{1/2} = \left[\frac{1}{g(1-\omega_0)} \right]^{1/2}$$

$$\begin{aligned} \tau_{\text{eff}} &= 2 \left[\frac{2\beta\omega_0}{1-\omega_0} \right]^{1/2} (1-\omega_0)\tau_1 = (8\beta\omega_0)^{1/2} (1-\omega_0)^{1/2} \tau_1 \\ &= \left[8 \frac{1}{2.5} (1-\omega_0) \right]^{1/2} \tau_1 = 2 \left[\frac{1-\omega_0}{g} \right]^{1/2} \tau_1 \end{aligned}$$

$$\therefore \tau = \frac{4u}{u^2 e^{-\tau_{\text{eff}}}}$$

$$= \frac{4}{u} e^{-\tau_{\text{eff}}}$$

$$\tau = 4 [g(1-\omega_0)]^{1/2} e^{-2[(1-\omega_0)/g]^{1/2} \tau_1}$$

which makes perfect sense. $\tau = 1/\mu_0$ (see below)
 for the 2 stream approximation; τ is the avg. probability
 of absorption per unit τ_1 is $1-\omega_0$, and $(1-\omega_0)^{1/2}$ is
 the result of a photon random walk.

Suppose $g = 7.5$, and $1-\omega_0 = 10^{-4}$.

$$\therefore \tau = 4 [7.5 \times 10^{-4}]^{1/2} e^{-2[10^{-4}/7.5]^{1/2} \tau_1}$$

$$= 1.1 \times 10^{-1} e^{-2[13.3 \times 10^{-6}]^{1/2} \tau_1}$$

$$= 0.11 e^{-7.2 \times 10^{-3} \tau_1}$$

Now the approximation is valid only when $e^{-\tau_{\text{eff}}} \gg 1$,

or $e^{-\tau_{\text{eff}}} \ll 1$. \therefore when $\tau_1, 7.2 \times 10^{-3} \tau_1 \gg 1$, or when

$$\tau_1 \gg \frac{1}{7.2 \times 10^{-3}} = 140$$

Note that for $\tau_1 \sim 300$,

$$\tau = 0.11 e^{-2.2} = 0.11 \times 0.12 = 1.3 \times 10^{-2}$$

\therefore it is possible to have enormous optical depths
 and still have a significant fraction of the light
 pass through if the particles are ^(small) forward-scatter. (18)

Now consider the case of $e^{\tau_{\text{eff}}} \gg 1$ — i.e., τ_{eff} is large optical depth — but with no scattering — then $\rho_0 = 1 - \rho_0$.

$$\therefore \tau \approx \frac{4u}{(u+1)^2} e^{-\tau_{\text{eff}}}$$

Note we may, in general, write

$$u = \left[\frac{(1-\rho_0) + \rho_0/g}{1-\rho_0} \right]^{1/2}$$

$$\tau_{\text{eff}} = 2u(1-\rho_0)\tau_1$$

If ρ_0 is not very close to unity, u will not be very small, and $\log \tau$ will imply $\log \tau_{\text{eff}}$.

E.g., $\rho_0 = 0.5$, $g = 3$, $u = \left[2 \left(0.5 + \frac{0.5}{3} \right) \right]^{1/2} =$

$$\left[2(0.66) \right]^{1/2} = \left[1.32 \right]^{1/2} = 1.15. \quad \tau_{\text{eff}} = 1.15 \tau_1, \quad \tau \approx e^{-1.15 \tau_1}$$

Note as $g \rightarrow \infty$, $u \rightarrow 1$.

Next consider pure absorption, $\rho_0 = 0$ \therefore $u = 1$,

$$\tau_{\text{eff}} = 2\tau_1, \text{ and}$$

$$\tau = \frac{4}{4 e^{\tau_{\text{eff}}}}$$

$$\therefore \tau = e^{-2\tau_1}$$

In the two-stream approximation the absorption

is effective averaging over μ $\tau_{\text{eff}} = \frac{1}{2}$.

Finally, consider the case that $\tau_{\text{eff}} = 1$.

$$\therefore \mathcal{F} = \frac{4u}{(u+1)^2 (1 + \tau_{\text{eff}}) - (u-1)^2 (1 - \tau_{\text{eff}})}$$
$$= \frac{4u}{\cancel{u^2 + 2u + 1} + u^2 \tau_{\text{eff}} + 2u \tau_{\text{eff}} + \tau_{\text{eff}} - \cancel{u^2 + 2u - 1} - u^2 \tau_{\text{eff}} + 2u \tau_{\text{eff}} - \tau_{\text{eff}}}$$

$$= \frac{4u}{4u + 4u \tau_{\text{eff}}}$$

$$\mathcal{F} = \frac{1}{1 + \tau_{\text{eff}}} = 1 - \tau_{\text{eff}} = e^{-\tau_{\text{eff}}}$$

For \mathcal{F} applications, reconsider the case

that $1 - \tau_0 < 1$. If this is true it is still possible, significant penetration of sunlight is possible. [We are here considering light incident from above the clouds, & neglect effects of surface albedo.] But as we go into the near infrared and τ_0 becomes smaller — although still very close to 1 — the incoming value of τ_0 begins to dominate, and near infrared radiation penetrates through. This is just the case we desire for a greenhouse effect. $\tau_{\text{eff}} \approx 100$ in this is significant result.

Now we return to the question of 3.75μ transparency of the ϕ clouds. In order for the hot underlying atmosphere to make negligible contribution to the 3.75μ brightness temp., we require

$$\tau(3.75 \mu) < 3 \times 10^{-3}.$$

We consider the clouds to have τ_{eff} , but we do not specify $\tau_0(3.75 \mu)$ as ^{usually} very close to unity. The transparency is (p. 17),

$$\tau = \frac{4u}{(u+1)^2} e^{-\tau_{\text{eff}}}.$$

We first estimate the coefficient τ . For $\tau_0 < 0.9$, $\frac{4u}{(u+1)^2} \sim 1$.

$$\text{For } 1 - \tau_0 < 1, u = \left[\frac{\tau_0/g}{1 - \tau_0} \right]^{1/2}, (u+1)^2 \approx \frac{\tau_0/g}{1 - \tau_0},$$

$$\frac{4u}{(u+1)^2} = 4 \left(\frac{1 - \tau_0}{\tau_0/g} \right)^{1/2}. \text{ For } 1 - \tau_0 = 10^{-2}, \frac{4u}{(u+1)^2} \approx 4 \sqrt{g} \times 10^{-1};$$

$$\text{for } 1 - \tau_0 = 10^{-3}, \frac{4u}{(u+1)^2} \approx 4 \sqrt{g} \times 10^{-2}. \text{ With } g \approx 4, \frac{4u}{(u+1)^2} \sim 1$$

$$\text{for } 1 - \tau_0 > 10^{-2}, \text{ For } 1 - \tau_0 \sim 10^{-2}, \frac{4u}{(u+1)^2} \sim 10^{-1}.$$

Smaller values of $1 - \tau_0$ at 3.75μ are not expected (see below). \therefore our condition on transparency requires

$$e^{-\tau_{\text{eff}}} < 2 \times 10^{-3}$$

$$\therefore \tau_{\text{eff}} > 6.2.$$

$$\therefore 2 [(1-\rho_0) + \rho_0/g]^{1/2} (1-\rho_0)^{1/2} \tau_1 > 6.2.$$

If we are specifying τ_1 from observation at some other wavelength, we will have a condition relating ρ_0 and g . The mean infrared albedo is $A_c \approx 0.75$; and between 0.7 and 1.7μ , the value is $A_c \approx 0.3$. This is a region where most (even strongly absorbing) materials absorb very poorly, and we can make conservative multiple scattering estimates. The fact of absorption gives us an upper limit on τ_1 :

$$A_c = 1 - \tau \leq 1 - \frac{1.25}{(\tau_1/g) + 1.4} = 0.9$$

$$\therefore 1.25 = 0.2 [(\tau_1/g) + 1.4]$$

$$\therefore 6.25 = (\tau_1/g) + 1.4$$

$$(\tau_1/g) = 4.85$$

$\therefore (\tau_1/g) \leq 4.9$, where g here is $g(1 \mu)$ and is probably less than $g(3.75 \mu)$.

$$\therefore 2 [(1-\rho_0) + \rho_0/g(3.75 \mu)]^{1/2} (1-\rho_0)^{1/2} \times 4.9 g(1 \mu) > 6.2$$

$$\therefore [(1-\rho_0) + \rho_0/g(3.75 \mu)]^{1/2} (1-\rho_0)^{1/2} g(1 \mu) > 0.63$$

For dust particles, $g(1\mu) \approx 2$; $g(3.75\mu) \approx 2$.

For water droplets [size range?], $g(1\mu) \approx 8$. For

3.75μ , the value of g depends on size of droplet,
+ we cover two cases, $g(3.75\mu) = 6$ and $g(3.75\mu) = 1$.

For dust,

$$[(1-\theta_0) + \frac{1}{2}\theta_0]^{1/2} (1-\theta_0)^{1/2} \times 2 > 0.63$$

$$[1 - \theta_0/2]^{1/2} (1-\theta_0)^{1/2} > 0.32$$

$$\text{For } \theta_0 = 0.8, (1 - \theta_0/2)^{1/2} (1 - \theta_0)^{1/2} = (1 - 0.4)^{1/2} (1 - 0.8)^{1/2}$$

$$= 0.6^{1/2} \cdot 0.2^{1/2} = 0.777 \times 0.446 = 0.35, \text{ which is, indeed, } > 0.32.$$

$$\therefore 1 - \theta_0 > 0.2$$

Had we chosen $g(3.75\mu) \approx 1$, we would have reached
about the same conclusion. Thus, for dust particles

to explain both the 3.75μ opacity, and the
 1μ albedo, they must have $\theta_0 < 0.8$ at 3.75μ ;
i.e., they must be ^{fairly} good absorbers at 3.75μ .

Note that had τ_1 been smaller (because of some
 1μ absorption), $1 - \theta_0$ would have been somewhat larger,

+ θ_0 even smaller.

For water droplets,

$$[(1 - \theta_0) + \theta_0/6]^{1/2} (1 - \theta_0)^{1/2} \times 6 > 0.63$$

$$\therefore (1 - 0.83\theta_0)^{1/2} (1 - \theta_0)^{1/2} > 0.079$$

$$\text{For } \theta_0 = 0.95, (1 - 0.83\theta_0)^{1/2} (1 - \theta_0)^{1/2} = (1 - 0.79)^{1/2} (1 - 0.95)^{1/2}$$

$$= (0.21)^{1/2} (0.05)^{1/2} = 0.46$$

$$0.22 = 0.10 \text{ which is } > 0.079.$$

$$\therefore 1 - \theta_0 > 5 \times 10^{-2}$$

For $g(3.75\mu) = 1$, we would have found

$$1 - \theta_0 > 1 \times 10^{-2}$$

Now we wish to compare these restrictions on $1 - \theta_0$ with the actual values for down particles of the appropriate size and composition, and check for consistency.

From a ref. in Neiberg's, we find that for indices of refraction of interest, 1.3 to 1.5 or so,

$$1 - \theta_0 \approx \frac{1}{3} (1 - e^{-\frac{2}{3}k_0 a})$$

$$\text{or } \theta_0 = e^{-\frac{2}{3}k_0 a}$$

Except for the factor $\frac{1}{3}$ which arises from spherical geometry of the particle, the absorption is simply being written in terms of Beer's law.

a is the particle radius, and Q is the absorption coefficient per unit thickness of the absorber. For H_2O , k_0 is listed in Dorey, Appendix 13. For some dust ~~and~~ ~~constituents~~ in an ~~Inst. Phys.~~ ~~compilation~~.

For H_2O ,

λ	k_0	$1 - E_0 (a = 1 \mu)$	$1 - E_0 (a = 3 \mu)$
0.5 μ	10^{-4} cm^{-1}	1.3×10^{-8}	4×10^{-8}
0.7 μ	10^{-2} cm^{-1}	$\sim 10^{-6}$	$\sim 10^{-6}$
1 μ	0.2	2.7×10^{-4}	8×10^{-5}
1.5 μ	20	1.4×10^{-3}	4×10^{-3}
2.7 μ	3000	0.3	0.5
3.5 μ	1000	8×10^{-2}	0.15

At 3.75μ the effective value of a may be larger than 1μ , because the ^{more} smaller particles in the distribution go to ∞ + 0 + 2 . \therefore a distrib. of sizes of water droplets in the μ -size range can explain the ~~spec.~~ ~~reflectivity~~ observed.

Dust, on the other hand, gives values for $\bar{a} \approx 5 \mu$ of

<u>Material</u>	<u>$1 - \bar{D}_0$</u>
Quartz	9×10^{-4}
Sapphire	7×10^{-6}
Calcite	7×10^{-3}

all far less than the $1 - \bar{D}_0$ (3.75μ) > 0.2 required. These particles, at least, fail by several orders of magnitude to satisfy the observational requirements. For calcite, the closest fit, to match these requirements, we require $\bar{a} \approx 150 \mu$, a value completely inadmissible on polarimetric grounds: According to v. der Hulst the position of the pos. maximum is very sensitive to particle size.

Again consider variation of k with θ_0 :

$$k \approx \frac{4u}{(u+1)^2} e^{-\tau_{\text{eff}}}$$

$$\tau_{\text{eff}} = 2 [(1-\theta_0) + \theta_0/g]^{1/2} (1-\theta_0)^{1/2} \tau_1$$

Now when absorption is very, very, $1-\theta_0 \ll 1$,

$$k \approx 4 [g(1-\theta_0)]^{1/2} e^{-2[(1-\theta_0)/g]^{1/2} \tau_1}$$

and for small enough $1-\theta_0$, k is independent of

τ_1 : the transmission (and albedo) is controlled only

by the single scattering albedo. As $1-\theta_0$ becomes

larger, k increases only as $(1-\theta_0)^{1/2}$ until,

quite abruptly, the exponential term

becomes important, and the optical depth

will begin to control the albedo. At that

pt. the albedo should decline abruptly.

The condition for τ_1 to become important is then

$$2 [(1-\theta_0) + \theta_0/g]^{1/2} (1-\theta_0)^{1/2} \tau_1 \sim 1.$$

This will still be true for $1-\theta_0 \ll 1$, so

$$2 [(1-\theta_0)/g]^{1/2} \tau_1 \sim 1$$

$$+ \frac{1-\theta_0}{g} \tau_1^2 \sim 1$$

$$1 - \theta_0 \approx \frac{g}{4\tau^2}$$

We have found for z , $[\tau, g(1\mu)] \approx 4.9$.

$$\therefore (1 - \theta_0) \approx \frac{g}{4 [\approx 4.9 g(1\mu)]^2}$$

$$\therefore (1 - \theta_0) < \frac{1 \times 10^{-2} g(\lambda)}{[g(1\mu)]^2}$$

Now for dust, $g(\lambda) \approx g(1\mu) \approx 2$, and

$$(1 - \theta_0)_{\text{dust}} < \text{~~10^{-2}}~~ 5 \times 10^{-3}$$

The corresponding wavelengths for dust are

Material λ_{crit} for τ_1 important

Calcite
Quartz
Sapphire

For water, $g(1\mu) \approx 8$, $g(\lambda)$ we must obtain by iteration, but roughly, $g(\lambda)$ is between 8 and 1.

$$\therefore 1 - \theta_0 < \frac{10^{-2}}{64 \times 3} = 1.9 \times 10^{-5}$$

This corresponds to $\lambda_{\text{crit}} \approx 1.5 \mu$ for $a = 1 \mu$ to the observed

1.5μ . Smaller a gives longer λ_{crit} . If we could

neglect $g(1\mu)$, $1 - \theta_0$ would be up by a factor of ≈ 10 ,

and $\lambda_{\text{crit}} = 1.5 \mu$. But why?

For $g(\lambda_{\text{crit}}) = 8$,

$$1 - \tau_0 < \frac{10^{-2}}{8} = 1.3 \times 10^{-3},$$

corresponding to $\lambda_{\text{crit}} = 1.2 \mu$ for 1μ water droplets.

For $g(\lambda_{\text{crit}}) = 1$,

$$1 - \tau_0 < \frac{10^{-2}}{6.4} = 1.6 \times 10^{-4},$$

corresponding to $\lambda_{\text{crit}} = 1.0 \mu$ for 1μ water droplets.

From Sato's photometry, a rather abrupt albedo decline occurs in the $\lambda > 1.2 \mu$ region, which we have attributed

to τ_1 .